The Mean and Variance Of The Sum Of Two Correlated Normally-Distributed Random Variates

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May, 2010

We are tasked with determining the mean and variance of the sum of 3 (a constant) times x, a normally-distributed random variate with mean 100 and standard deviation 50, and 5 (a constant) times y, a normally-distributed random variate with mean 60 and standard deviation 20. The correlation between x and y is 0.60.

Legend of Symbols

A	=	Multiple of x
B	=	Multiple of y
x	=	Normally-distributed random variate with mean μ_x and variance σ_x^2
y	=	Normally-distributed random variate with mean μ_y and variance σ_y^2
ρ	=	Pearson correlation coefficient between x and y
z_x	=	Normally-distributed random variate with mean zero and variance one - Used in the equation of x
z_y	=	Normally-distributed random variate with mean zero and variance one - Used in the equation of y
z	=	Normally-distributed random variate with mean zero and variance one - Used in both equations of x and y
θ	=	$\sqrt{\rho}$
ϕ	=	$\sqrt{1-\rho}$

The Equations for Ax and By

We can write the equations for x and y as follows...

$$x = \mu_x + \theta\sigma_x z + \phi\sigma_x z_x \tag{1}$$

$$y = \mu_y + \theta\sigma_y z + \phi\sigma_y z_y \tag{2}$$

The two equations above reflect the fact that x and y are correlated.

The first moment of the distribution of $Ax + By$ (see Appendix B) is...

$$\mathbb{E}\Big[Ax + By\Big] = A\mu_x + B\mu_y \tag{3}$$

The second moment of the distribution of $Ax + By$ (see Appendix C) is...

$$\mathbb{E}\Big[(Ax + By)^2\Big] = A^2\mu_x^2 + B^2\mu_y^2 + 2AB\mu_x\mu_y + 2AB\theta^2\sigma_x\sigma_y + A^2\theta^2\sigma_x^2 + A^2\phi^2\sigma_x^2 + B^2\theta^2\sigma_y^2 + B^2\phi^2\sigma_y^2 \tag{4}$$

The mean of the distribution of $Ax + By$ is...

$$\begin{aligned} mean_{Ax+By} &= \mathbb{E}\Big[Ax + By\Big] \\ &= A\mu_x + B\mu_y \end{aligned} \tag{5}$$

The variance of the distribution of $Ax + By$ is...

$$\begin{aligned} var_{Ax+By} &= \mathbb{E}\Big[(Ax+By)^2\Big] - \Big[\mathbb{E}\Big[Ax+By\Big]\Big]^2 \\ &= A^2\mu_x^2 + B^2\mu_y^2 + 2AB\mu_x\mu_y + 2AB\theta^2\sigma_x\sigma_y + A^2\theta^2\sigma_x^2 + A^2\phi^2\sigma_x^2 + B^2\theta^2\sigma_y^2 + B^2\phi^2\sigma_y^2 - (A\mu_x + B\mu_y)^2 \\ &= 2AB\theta^2\sigma_x\sigma_y + A^2\theta^2\sigma_x^2 + A^2\phi^2\sigma_x^2 + B^2\theta^2\sigma_y^2 + B^2\phi^2\sigma_y^2 \\ &= A^2\sigma_x^2(\rho + 1 - \rho) + B^2\sigma_y^2(\rho + 1 - \rho) + 2AB\rho\sigma_x\sigma_y \\ &= A^2\sigma_x^2 + B^2\sigma_y^2 + 2AB\rho\sigma_x\sigma_y \end{aligned} \tag{6}$$

Problem Solution

The mean of $Ax + By$ per equation (5) above is...

$$\begin{aligned} mean &= (3)(100) + (5)(60) \\ &= 600 \end{aligned} \tag{7}$$

The variance of $Ax + By$ per equation (6) above is...

$$\begin{aligned} variance &= (3^2)(50^2) + (5^2)(20^2) + (2)(3)(5)(0.60)(50)(20) \\ &= 50,500 \end{aligned} \tag{8}$$

Appendix

A. Rules on expectations of random variables with mean zero and variance one:

$$\mathbb{E}\Big[z^1\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_x^1\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_y^1\Big] = 0$$

$$\mathbb{E}\Big[z^2\Big] = 1 \quad ; \quad \mathbb{E}\Big[z_x^2\Big] = 1 \quad ; \quad \mathbb{E}\Big[z_y^2\Big] = 1$$

$$\mathbb{E}\Big[z^3\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_x^3\Big] = 0 \quad ; \quad \mathbb{E}\Big[z_y^3\Big] = 0$$

$$\mathbb{E}\Big[z^4\Big] = 3 \quad ; \quad \mathbb{E}\Big[z_x^4\Big] = 3 \quad ; \quad \mathbb{E}\Big[z_y^4\Big] = 3$$

B. The expected value of $Ax + By$ is...

$$\begin{aligned} \mathbb{E}\Big[Ax+By\Big] &= \mathbb{E}\Big[A(\mu_x + \theta\sigma_x z + \phi\sigma_x z_x) + B(\mu_y + \theta\sigma_y z + \phi\sigma_y z_y)\Big] \\ &= \mathbb{E}\Big[A\mu_x + A\theta\sigma_x z + A\phi\sigma_x z_x + B\mu_y + B\theta\sigma_y z + B\phi\sigma_y z_y\Big] \\ &= A\mu_x + B\mu_y \end{aligned} \tag{9}$$

One equation with six variables yields $6^1 = 6$ permutations as follows...

Perm	Eq	Value		Result
01	a1	$A\mu_x$	=	$A\mu_x$
02	a2	$A\theta\sigma_x z$	=	0
03	a3	$A\phi\sigma_x z_x$	=	0
04	a4	$B\mu_y$	=	$B\mu_y$
05	a5	$B\theta\sigma_y z$	=	0
06	a6	$B\phi\sigma_y z_y$	=	0

C. The expected value of $Ax + By$ quantity squared is...

$$\begin{aligned}\mathbb{E}\Big[(Ax+By)^2\Big] &= \mathbb{E}\Big[(A\mu_x + A\theta\sigma_x z + A\phi\sigma_x z_x + B\mu_y + B\theta\sigma_y z + B\phi\sigma_y z_y)^2\Big] \\ &= A^2\mu_x^2 + B^2\mu_y^2 + 2AB\mu_x\mu_y + 2AB\theta^2\sigma_x\sigma_y + A^2\theta^2\sigma_x^2 + A^2\phi^2\sigma_x^2 + B^2\theta^2\sigma_y^2 + B^2\phi^2\sigma_y^2 \qquad (10)\end{aligned}$$

Two equations each six three variables yields $6^2 = 36$ permutations as follows...

Perm	Eq 1	Eq 2	Value 1	Value 2		Result
01	a1	b1	$A\mu_x$	$A\mu_x$	=	$A^2\mu_x^2$
02	a1	b2	$A\mu_x$	$A\theta\sigma_x z$	=	0
03	a1	b3	$A\mu_x$	$A\phi\sigma_x z_x$	=	0
04	a1	b4	$A\mu_x$	$B\mu_y$	=	$AB\mu_x\mu_y$
05	a1	b5	$A\mu_x$	$B\theta\sigma_y z$	=	0
06	a1	b6	$A\mu_x$	$B\phi\sigma_y z_y$	=	0
07	a2	b1	$A\theta\sigma_x z$	$A\mu_x$	=	0
08	a2	b2	$A\theta\sigma_x z$	$A\theta\sigma_x z$	=	$A^2\theta^2\sigma_x^2$
09	a2	b3	$A\theta\sigma_x z$	$A\phi\sigma_x z_x$	=	0
10	a2	b4	$A\theta\sigma_x z$	$B\mu_y$	=	0
11	a2	b5	$A\theta\sigma_x z$	$B\theta\sigma_y z$	=	$AB\theta^2\sigma_x\sigma_y$
12	a2	b6	$A\theta\sigma_x z$	$B\phi\sigma_y z_y$	=	0
13	a3	b1	$A\phi\sigma_x z_x$	$A\mu_x$	=	0
14	a3	b2	$A\phi\sigma_x z_x$	$A\theta\sigma_x z$	=	0
15	a3	b3	$A\phi\sigma_x z_x$	$A\phi\sigma_x z_x$	=	$A^2\phi^2\sigma_x^2$
16	a3	b4	$A\phi\sigma_x z_x$	$B\mu_y$	=	0
17	a3	b5	$A\phi\sigma_x z_x$	$B\theta\sigma_y z$	=	0
18	a3	b6	$A\phi\sigma_x z_x$	$B\phi\sigma_y z_y$	=	0
19	a4	b1	$B\mu_y$	$A\mu_x$	=	$AB\mu_x\mu_y$
20	a4	b2	$B\mu_y$	$A\theta\sigma_x z$	=	0
21	a4	b3	$B\mu_y$	$A\phi\sigma_x z_x$	=	0
22	a4	b4	$B\mu_y$	$B\mu_y$	=	$B^2\mu_y^2$
23	a4	b5	$B\mu_y$	$B\theta\sigma_y z$	=	0
24	a4	b6	$B\mu_y$	$B\phi\sigma_y z_y$	=	0
25	a5	b1	$B\theta\sigma_y z$	$A\mu_x$	=	0
26	a5	b2	$B\theta\sigma_y z$	$A\theta\sigma_x z$	=	$AB\theta^2\sigma_x\sigma_y$
27	a5	b3	$B\theta\sigma_y z$	$A\phi\sigma_x z_x$	=	0
28	a5	b4	$B\theta\sigma_y z$	$B\mu_y$	=	0
29	a5	b5	$B\theta\sigma_y z$	$B\theta\sigma_y z$	=	$B^2\theta^2\sigma_y^2$
30	a5	b6	$B\theta\sigma_y z$	$B\phi\sigma_y z_y$	=	0
31	a6	b1	$B\phi\sigma_y z_y$	$A\mu_x$	=	0
32	a6	b2	$B\phi\sigma_y z_y$	$A\theta\sigma_x z$	=	0
33	a6	b3	$B\phi\sigma_y z_y$	$A\phi\sigma_x z_x$	=	0
34	a6	b4	$B\phi\sigma_y z_y$	$B\mu_y$	=	0
35	a6	b5	$B\phi\sigma_y z_y$	$B\theta\sigma_y z$	=	0
36	a6	b6	$B\phi\sigma_y z_y$	$B\phi\sigma_y z_y$	=	$B^2\phi^2\sigma_y^2$